Proceedings of DETC'98
1998 ASME Design Engineering Technical Conferences
September 13-16, 1998, Atlanta, Georgia, USA

DETC98/MECH-5887

A CLASSIFICATION SCHEME FOR PLANAR 4R, SPHERICAL 4R, AND SPATIAL RCCC LINKAGES TO FACILITATE COMPUTER ANIMATION

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ABSTRACT

In this paper we develop a complete classification scheme for planar, spherical and spatial four-bar linkages. The goal of this classification is to note all of the subtleties of motion that an arbitrary set of four link lengths can define. A classification should exist only between the mechanisms that exhibit similar ranges of motion at both the input and output. In the planar case, three parameters being identified as either positive, negative, or zero are necessary to completely characterize all possible ranges of motion. In the spherical and spatial cases, four parameters being identified as positive, negative or zero are needed. The result is 27 classifications of planar mechanisms and 81 for spherical and spatial *RCCC* mechanisms.

INTRODUCTION

The synthesis and analysis of planar four-bar mechanisms via software utilizing interactive graphics is a practice that has now existed for thirty years. A few of the programs developed for this purpose include KINSYN (Kaufman, 1978), RECSYN (Chuang et al., 1981) and LINCAGES (Erdman and Gustafson, 1977). For the synthesis and analysis of spherical four-bar mechanisms, Larochelle et al. (1993) have developed *S*PHINX. Of particular use in the analysis of a mechanism is the animation of that mechanism through some appropriate range of motion. One possible range is all motions achievable by a given mechanism.

Classification schemes for planar four-bar mechanisms can be found in almost any text on the subject of machine theory. For example, see Erdman and Sandor (1997) or Norton (1992). The gross classification of a mechanism is Grashof (having a fully rotatable link) or non-Grashof and then it is further categorized with the familiar titles of crank-rocker, Grashof double-rocker, drag link and the likes. Proofs of the Grashof criterion can be found in Williams and Reinholtz (1986), Paul (1979), and the extension to spatial *RSSR* four-bar linkages in Kazerounian and Solecki (1993). For most planar mechanism analysis, these standard classification methods are ideal. Although a mechanism's classification is not always necessary for its animation over some range of motion, the classification helps to expedite the determination of the range and, occasionally, can be necessary. The necessity for a thorough classification scheme for mechanisms arises when the mechanism is classified as a "change-point mechanism", or lying in the region that separates Grashof mechanisms from those that are non-Grashof. There exists a diverse array of these change-point mechanisms to be differentiated amongst. In addition, mechanisms of this type are commonly encountered when performing solution rectification on large sets of candidate mechanisms. For example, *S*PHINX generates a discrete representation of the ∞^2 spherical four-bar mechanisms which will guide a moving body through four orientations in space. This discretized solution space is presented to the user as a *linkage type map* (Ruth and McCarthy, 1997 and Murray and

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McCarthy, 1995). The linkage type map color codes each solution according to its mechanism type and at the boundaries of the familiar mechanism types these change-point mechanisms must occur.

Spherical mechanisms admit a classification scheme similar to that of planar mechanisms (see Duffy, 1980 and Chiang, 1988). In addition, Grashof's law holds in a modified form for the spherical case. The primary utility of these schemes for the spherical case is to draw a comparison with their planar counterparts to allow the use of intuition developed about planar four-bars.

The scheme developed here, at least for the planar case, defines parameters similar to those determined by Bottema and Roth (1979) for classification of the image curves of planar four-bar motion. Two works of note that seek more complete identifications of the sets of all four-bars are Barker's (1985) comprehensive classification of planar four-bar mechanisms and Savage and Hall's (1970) similar treatment of spherical four-bars (including a discussion by Soni).

PLANAR MECHANISM ANALYSIS

Consider the planar mechanism shown in Fig. 1 The relationship between the input angle Θ of the driving link to the output angle Ψ of the output link is

$$\Psi(\Theta) = \arctan\left(\frac{B}{A}\right) \pm \arccos\left(\frac{C}{\sqrt{A^2+B^2}}\right) \quad (1)$$

where

$$\begin{aligned} A(\Theta) &= 2ab\cos\Theta - 2gb, \\ B(\Theta) &= 2ab\sin\Theta, \text{ and} \\ C(\Theta) &= g^2+b^2+a^2-h^2-2ag\cos\Theta. \end{aligned} \quad (2)$$

Note that the arctan() function in Eq. 1 must identify angles on all four quadrants to be accurate.

The argument of the arccosine term in Eq. 1 must be in the range -1 to +1 for a solution to exist. Therefore, $A(\Theta)^2+B(\Theta)^2-C(\Theta)^2 \geq 0$, and this relation defines the range of the angular movement of the input link. Expanding the inequality yields a quadratic equation in $\cos\Theta$ that has two roots.

$$C_1 = \frac{(g^2+a^2)-(h-b)^2}{2ag}, \quad (3)$$

$$C_2 = \frac{(g^2+a^2)-(h+b)^2}{2ag}. \quad (4)$$

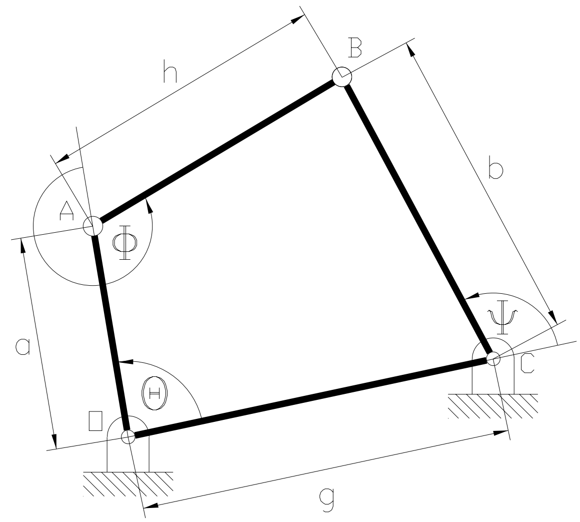

Figure 1. A PLANAR FOUR-BAR MECHANISM

These equations are the familiar cosine laws defining the angle Θ at the limits to the mechanism's range of motion. These limiting angles exist if $-1 < C_1, C_2 < 1$. There are three cases:

1. Neither of the limiting angles $\pm\Theta_i, i = 1,2$ exists, which means the input link fully rotates;
2. Only one of the two angles exists: a) if it is Θ_1 then the input link rocks through the angle $\Theta = \pi$ between $\pm\Theta_1$, and, b) if Θ_2 exists then the input link rocks through the angle $\Theta = 0$ between $\pm\Theta_2$;
3. Both angles exist, which means the input link rocks between Θ_1 and Θ_2 and between $-\Theta_1$ and $-\Theta_2$ and does not pass through either 0 or π.

The Input Link

The root C_1 determines the smallest positive angle the driving link can reach. The link can reach $\Theta = 0$ if

$$\frac{(g^2+a^2)-(h-b)^2}{2ag} \geq 1, \quad (5)$$

or,

$$(g-a)^2 \geq (h-b)^2. \quad (6)$$

Introduce the parameters

$$T_1 = g-a+h-b \text{ and } T_2 = g-a-h+b. \quad (7)$$

Noting that $T_1T_2 = (g-a)^2-(h-b)^2$, the driving link passes through the angle $\Theta = 0$ if the product $T_1T_2 \geq 0$.

The root C_2 determines the largest positive angle reachable by the driving link. The range of movement of the driving link includes $\Theta = \pi$ if

$$\frac{(g^2+a^2)-(h+b)^2}{2ag} \leq -1, \tag{8}$$

which simplifies to

$$(g+a)^2 \leq (h+b)^2. \tag{9}$$

Since all of the link lengths are positive we need only consider the parameter

$$T_3 = h+b-g-a. \tag{10}$$

The condition $T_3 \geq 0$ identifies that the input link passes through $\Theta = \pi$.

The three parameters, $T_i, i = 1,2,3$, characterize the movement of the driving link:

1. The driving link fully rotates: $T_1T_2 \geq 0$, and $T_3 \geq 0$;
2. The driving link rocks through $\Theta = 0$: $T_1T_2 \geq 0$ and $T_3 < 0$;
3. The driving link rocks through $\Theta = \pi$: $T_1T_2 < 0$ and $T_3 \geq 0$; and
4. The driving link rocks over two ranges neither of which includes 0 or π: $T_1T_2 < 0$ and $T_3 < 0$.

The Output Link

The limiting values of $\cos\Psi$ associated with the output link of a planar mechanism are given by:

$$C_3 = \frac{(h+a)^2-(g^2+b^2)}{2bg}, \tag{11}$$

$$C_4 = \frac{(h-a)^2-(g^2+b^2)}{2bg}. \tag{12}$$

The condition that the output link pass through $\Psi = 0$ is obtained from C_3 as

$$\frac{(h+a)^2-(g^2+b^2)}{2bg} \geq 1, \tag{13}$$

or,

$$(h+a)^2 \geq (g+b)^2. \tag{14}$$

Identify the parameters

$$h+a-g-b = -T_2, \tag{15}$$

which leads to the result that if $T_2 \leq 0$ the link passes through zero, and if $T_2 > 0$ it does not.

The output link passes through $\Psi = \pi$ if

$$\frac{(h-a)^2-(g^2+b^2)}{2bg} \leq -1, \tag{16}$$

or,

$$(h-a)^2 \leq (g-b)^2. \tag{17}$$

Using the parameters

$$g-b+h-a = T_1 \text{ and } g-b-h+a = -T_3, \tag{18}$$

if $T_1T_3 \leq 0$ then the link passes through π, otherwise it does not.

The result is that the same parameters, $T_i, i = 1,2,3$ characterize the movement of the output link, and we have the four cases:

1. The output link fully rotates: $T_2 \leq 0$ and $T_1T_3 \leq 0$;
2. The output link rocks through $\Psi = 0$: $T_2 \leq 0$ $T_1T_3 > 0$;
3. The output link rocks through $\Psi = \pi$: $T_2 > 0$ and $T_1T_3 \leq 0$; and
4. The output link rocks over two ranges: $T_2 > 0$ and $T_1T_3 > 0$.

PLANAR MECHANISM CLASSIFICATION

The three parameters $T_i, i = 1,2,3$ classify the movement of the driving and output links of a $4R$ linkage into eight basic types.

If a configuration exists such that all four joints of a planar linkage lie on a line the mechanism is said to "fold." If one (or more) of the characteristics $T_i, i = 1,2,3$ is zero, then the mechanism is a *foldable linkage*. If we consider the parameters $T_i, i = 1,2,3$ can take the values $(+,0,-)$, then there are 27 classifications of planar $4R$ linkages, 19 of which fold. The number of parameters T_i that are zero equals the number of folding configurations of the linkage.

Grashof's Condition

Grashof's condition states that one of the links in a mechanism can fully rotate if the sum of the lengths of the longest and shortest links is less than (or equal to) the sum of the

Table 1. BASIC PLANAR $4R$ LINKAGE TYPES

	Linkage type	T_1	T_2	T_3
1.	Crank-rocker	+	+	+
2.	Rocker-crank	+	-	-
3.	Double-crank	-	-	+
4.	Grashof double-rocker	-	+	-
5.	00 double-rocker	-	-	-
6.	0π double-rocker	+	+	-
7.	$\pi 0$ double-rocker	+	-	+
8.	$\pi\pi$ double-rocker	-	+	+

lengths of the two intermediate links. The classification presented here leads to an alternate expression for Grashof's condition. Note from Table 1 that the mechanism contains a fully rotatable link(and is not a change-point mechanism) if $T_1 T_2 T_3 > 0$. This quantity is a function of the link lengths a, b, g and h where the identification of the longest and shortest links is unnecessary to determine whether or not the mechanism contains a fully rotatable link.

An Example

Consider a mechanism A with link lengths $\{a,b,g,h\}$ = $\{1,3,5,3\}$ and a mechanism B with $\{a,b,g,h\}$ = $\{3,4,3,2\}$. The associated parameters are, for mechanism A, $\{T_1,T_2,T_3\}$ = $\{4,4,0\}$ and, for B, $\{T_1,T_2,T_3\}$ = $\{-2,2,0\}$. Both mechanisms are readily classified as simple folding four-bars. In fact, both fold (all of the pivots become colinear) at the location where the input angle $\Theta = \pi$. This is where the similarities end, however, with the input link on mechanism A being fully rotatable and the input link on B rocking through $\Theta = \pi$. An extension of Table 1 to include all 27 cases could be used to rapidly identify this, noting that any $\{+,+,0\}$ mechanism has a fully rotatable input crank and any $\{-,+,0\}$ mechanism's input link rocks through $\Theta = \pi$.

SPHERICAL MECHANISM ANALYSIS

Consider the mechanism shown in Fig. 2. The relationship between the input angle Θ of the driving link to the output angle Ψ of the output link is

$$\Psi(\Theta) = \arctan\left(\frac{B}{A}\right) \pm \arccos\left(\frac{C}{\sqrt{A^2+B^2}}\right) \tag{19}$$

where

$$A(\Theta) = \sin\alpha\sin\beta\cos\gamma\cos\Theta - \cos\alpha\sin\beta\sin\gamma,$$

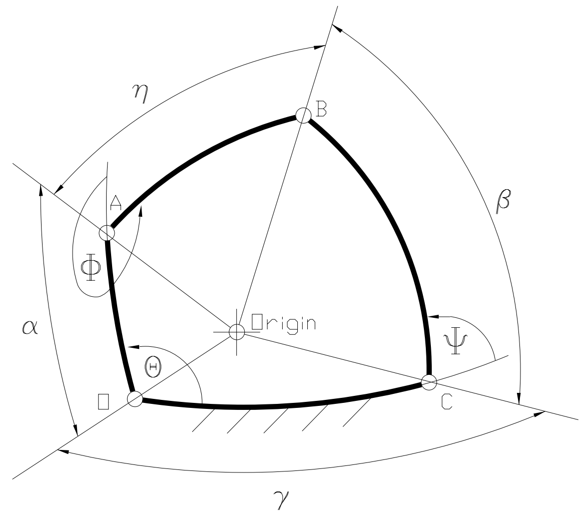

Figure 2. A SPHERICAL FOUR-BAR MECHANISM

$$\begin{aligned} B(\Theta) &= \sin\alpha\sin\beta\sin\Theta, \text{ and} \\ C(\Theta) &= \cos\eta - \cos\alpha\cos\beta\cos\gamma \\ &\quad - \sin\alpha\cos\beta\sin\gamma\cos\Theta \end{aligned} \tag{20}$$

and $0 \le \alpha, \beta, \eta, \gamma \le \pi$.

The argument of the arccosine term in Eq. 19 must be in the range -1 to +1 for a solution to exist. Therefore, $A(\Theta)^2 + B(\Theta)^2 - C(\Theta)^2 \ge 0$, and this relation defines the range of the angular movement of the input link. Expanding the inequality yields a quadratic equation in $\cos\Theta$ that has two roots

$$C_1 = \frac{\cos(\eta-\beta) - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}, \tag{21}$$

$$C_2 = \frac{\cos(\eta+\beta) - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}. \tag{22}$$

These equations are the spherical cosine laws defining the angle Θ at the limits to its range of motion. These limiting angles exist if $-1 < C_1, C_2 < 1$. There are three cases:

1. Neither of the limiting angles $\pm\Theta_i, i = 1,2$ exists, which means the input link fully rotates;
2. Only one of the two angles exists: a) if it is Θ_1 then the input link rocks through the angle $\Theta = \pi$ between $\pm\Theta_1$, and, b) if Θ_2 exists then the input link rocks through the angle $\Theta = 0$ between $\pm\Theta_2$;
3. Both angles exist, which means the input link rocks between Θ_1 and Θ_2 and between $-\Theta_1$ and $-\Theta_2$ and does not pass through either 0 or π.

The Input Link

The root C_1 determines the smallest positive angle the driving link can reach. The link can reach $\Theta = 0$ if

$$\frac{\cos(\eta-\beta)-\cos\alpha\cos\gamma}{\sin\alpha\sin\gamma} \geq 1, \tag{23}$$

or,

$$\cos(\eta-\beta) \geq \cos(\gamma-\alpha). \tag{24}$$

The combinations of angular lengths that satisfy this relation are

$$|\gamma-\alpha| \geq |\eta-\beta|. \tag{25}$$

Introduce the parameters

$$T_1 = \gamma-\alpha+\eta-\beta \text{ and } T_2 = \gamma-\alpha-\eta+\beta. \tag{26}$$

The driving link passes through the angle $\Theta = 0$ if the product $T_1T_2 \geq 0$.

The root C_2 determines the largest positive angle reachable by the driving link. The range of movement of the driving link includes $\Theta = \pi$ if

$$\frac{\cos(\eta+\beta)-\cos\alpha\cos\gamma}{\sin\alpha\sin\gamma} \leq -1, \tag{27}$$

which simplifies to

$$\cos(\eta+\beta) \leq \cos(\gamma+\alpha). \tag{28}$$

Since all of the link lengths are in the range 0 to π, this condition is equivalent to

$$|\pi-(\eta+\beta)| \leq |\pi-(\gamma+\alpha)|. \tag{29}$$

Define the parameters

$$T_3 = \eta+\beta-\gamma-\alpha \text{ and } T_4 = 2\pi-\eta-\beta-\gamma-\alpha. \tag{30}$$

The condition $T_3T_4 \geq 0$ identifies that the input link passes through $\Theta = \pi$.

The four parameters, $T_i, i = 1,2,3,4$, characterize the movement of the driving link:

1. The driving link fully rotates: $T_1T_2 \geq 0$, and $T_3T_4 \geq 0$;
2. The driving link rocks through $\Theta = 0$: $T_1T_2 \geq 0$ and $T_3T_4 < 0$;
3. The driving link rocks through $\Theta = \pi$: $T_1T_2 < 0$ and $T_3T_4 \geq 0$; and
4. The driving link rocks over two ranges neither of which includes 0 or π: $T_1T_2 < 0$ and $T_3T_4 < 0$.

The Output Link

The limiting values of $\cos\Psi$ associated with the output link of a spherical mechanism are given by:

$$C_3 = \frac{\cos\gamma\cos\beta-\cos(\eta+\alpha)}{\sin\gamma\sin\beta}, \tag{31}$$

$$C_4 = \frac{\cos\gamma\cos\beta-\cos(\eta-\alpha)}{\sin\gamma\sin\beta}. \tag{32}$$

The condition that the output link pass through $\Psi = 0$ is obtained from C_3 as

$$\frac{\cos\gamma\cos\beta-\cos(\eta+\alpha)}{\sin\gamma\sin\beta} \geq 1, \tag{33}$$

or,

$$\cos(\eta+\alpha) \leq \cos(\gamma+\beta). \tag{34}$$

This condition can be expressed as

$$|\pi-(\eta+\alpha)| \leq |\pi-(\gamma+\beta)|. \tag{35}$$

Identify the parameters

$$\eta+\alpha-\gamma-\beta = -T_2 \text{ and } 2\pi-\eta-\alpha-\gamma-\beta = T_4, \tag{36}$$

which leads to the result that if $T_2T_4 \leq 0$ the link passes through zero, and if $T_2T_4 > 0$ it does not.

The output link passes through $\Psi = \pi$ if

$$\frac{\cos\gamma\cos\beta-\cos(\eta-\alpha)}{\sin\gamma\sin\beta} \leq -1, \tag{37}$$

or,

$$\cos(\eta-\alpha) \geq \cos(\gamma-\beta). \tag{38}$$

The link lengths that satisfy this condition are either

$$|\gamma-\beta| \geq |\eta-\alpha|. \quad (39)$$

Using the parameters

$$\gamma-\beta+\eta-\alpha = T_1 \text{ and } \gamma-\beta-\eta+\alpha = -T_3, \quad (40)$$

if $T_1T_3 \leq 0$ then the link passes through π, otherwise it does not.

The result is that the same parameters, $T_i, i = 1,2,3,4$ characterize the movement of the output link, and we have the four cases:

1. The output link fully rotates: $T_2T_4 \leq 0$ and $T_1T_3 \leq 0$;
2. The output link rocks through $\Psi = 0$: $T_2T_4 \leq 0$ $T_1T_3 > 0$;
3. The output link rocks through $\Psi = \pi$: $T_2T_4 > 0$ and $T_1T_3 \leq 0$; and
4. The output link rocks over two ranges: $T_2T_4 > 0$ and $T_1T_3 > 0$.

SPHERICAL MECHANISM CLASSIFICATION

The four parameters $T_i, i = 1,2,3,4$ classify the movement of the driving and output links of a $4R$ linkage into two sets of eight basic types denoted by those with $T_4 > 0$ and those with $T_4 < 0$. The eight spherical mechanisms with positive T_4 have the same properties as the planar $4R$ mechanisms with the same linkage type. The linkages with $T_4 < 0$ have link lengths that add up to greater than 2π and *wrap* around the sphere. The characteristics T_1, T_2, and T_3 for a linkage with $T_4 < 0$ are the negation of these characteristics for the same type of linkage with $T_4 > 0$. Thus two spherical linkages, each with the negative set of characteristics of the other, will have the same overall movement of the input and output links.

If a configuration exists such that all four joints of a spherical linkage lie on a plane the mechanism is said to "fold." If one (or more) of the characteristics $T_i, i = 1,2,3,4$ is zero, then the mechanism is a *foldable linkage*. If we consider the parameters $T_i, i = 1,2,3,4$ can take the values $(+,0,-)$, then there are 81 classifications of spherical $4R$ linkages, 65 of which fold. The number of parameters T_i that are zero equals the number of folding configurations of the linkage.

Grashof's Condition

For any given set of four link lengths defining a spherical mechanism, changing any two of the link lengths to their supplements defines a mechanism capable of the same motion. To apply Grashof's condition to spherical four-bar mechanisms, this rule must be applied to the link lengths to determine the set with the shortest total length. Grashof's condition can now be extended to the sphere: one of the links in a spherical mechanism fully rotates if the sum of the lengths of the longest and shortest links is less than (or equal to) the sum of the lengths of the two intermediate links. The classification presented here leads to an alternate expression for this condition. Note from Table 2 that a spherical mechanism contains a fully rotatable link(and is not a change-point mechanism) if $T_1T_2T_3T_4 > 0$. This quantity is a function of the link lengths α, β, γ and η where the identification of the longest and shortest links is unnecessary to determine whether or not the mechanism contains a fully rotatable link.

Table 2. BASIC SPHERICAL $4R$ LINKAGE TYPES

	Linkage type	T_1	T_2	T_3	T_4
1.	Crank-rocker	+	+	+	+
2.	Rocker-crank	+	-	-	+
3.	Double-crank	-	-	+	+
4.	Grashof double-rocker	-	+	-	+
5.	$00+$ double-rocker	-	-	-	+
6.	$0\pi+$ double-rocker	+	+	-	+
7.	$\pi0+$ double-rocker	+	-	+	+
8.	$\pi\pi+$ double-rocker	-	+	+	+
9.	Crank-rocker	-	-	-	-
10.	Rocker-crank	-	+	+	-
11.	Double-crank	+	+	-	-
12.	Grashof double-rocker	+	-	+	-
13.	$00-$ double-rocker	+	+	+	-
14.	$0\pi-$ double-rocker	-	-	+	-
15.	$\pi0-$ double-rocker	-	+	-	-
16.	$\pi\pi-$ double-rocker	+	-	-	-

SPATIAL MECHANISM ANALYSIS

Consider the spatial *RCCC* mechanism shown in Fig. 3 where the rotation of the revolute joint is considered the input to this linkage. Associated with each *RCCC* mechanism is a spherical image. The spherical image is a spherical four-bar mechanism with link lengths equal to the angular twist of the links of the *RCCC* mechanism, see Duffy (1980). Hence, by having previously developed a classification for spherical $4R$ linkages we can now classify spatial linkages. We classify spatial *RCCC* mechanisms according to the linkage type of their corresponding spherical image.

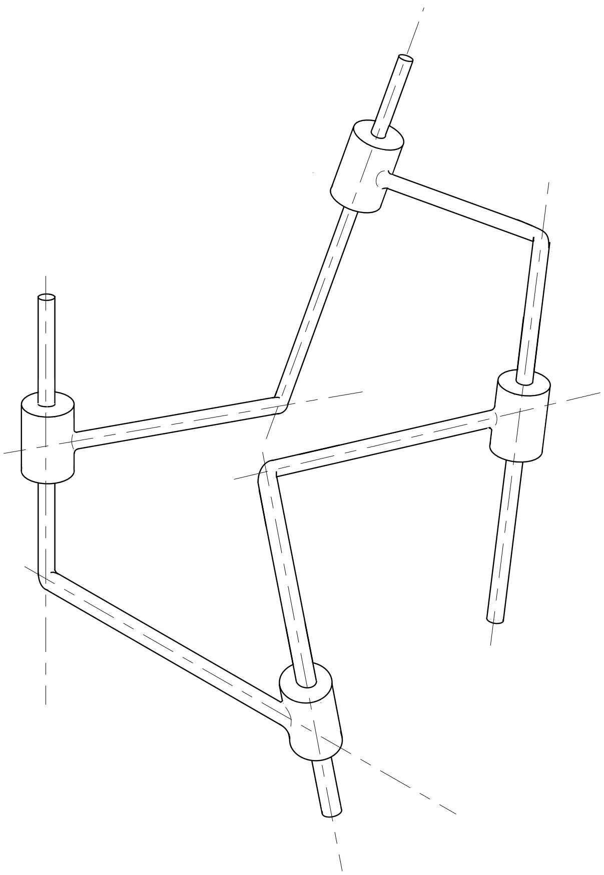

Figure 3. A SPATIAL *RCCC* FOUR-BAR MECHANISM

CONCLUSIONS

In this paper we have developed a complete classification scheme for planar and spherical 4*R* linkages. Moreover, we classify spatial *RCCC* linkages according to their associated spherical image. The goal of this classification is to note all of the subtleties of motion that an arbitrary set of four link lengths can define. The result is 27 unique classifications of planar mechanisms and 81 for spherical and spatial mechanisms.

ACKNOWLEDGMENTS

The support of the National Science Foundation (Grant #DMI-9612062) and the Oak Ridge Associated Universities Junior Faculty Enhancement Award are gratefully acknowledged by Dr. Larochelle.

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